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LINEAR AND NONLINEAR DYNAMICS OF CANTILEVERED CYLINDERS IN AXIAL FLOW. PART 2: THE EQUATIONS OF MOTION

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In this paper a nonlinear equation of motion is derived for the dynamics of a slender cantilevered cylinder in axial flow, generally terminated by an ogival free end. Inviscid forces are modelled by an extension of Lighthill's slender-body work to third-order accuracy. The viscous, hydrostatic and gravity-related terms are derived separately, to the same accuracy. The equation of motion is obtained via Hamilton's principle. The boundary conditions related to the ogival free end are also derived separately. The paper is concluded by a discussion of the methods used to obtain the solutions presented in Part 3 of this study.

1. INTRODUCTION

A general introduction to this three-part study into the dynamics of cantilevered cylinders in axial flow has been given in Part 1 (Païdoussis *et al*. 2002), wherein the physical aspects of the dynamics are considered.

The experimentally observed dynamical behaviour of the system makes it abundantly clear that it would be useful to have a theory available which would predict nonlinear, as well as linear, aspects of the observed behaviour. Linear aspects are mainly related to (i) the general behaviour of the system before its first loss of stability, e.g., whether motions are damped by the action of the flow, and (ii) the critical flow velocities for the bifurcations associated with changes in dynamical behaviour, e.g., the pitchfork bifurcation giving rise to divergence and the Hopf or other type of bifurcation giving rise to single- or coupled-mode flutter. However, several other, important aspects of dynamical behaviour can only be predicted via nonlinear theory; for example: (i) the existence of post-divergence bifurcations (e.g., the flutter predicted by linear theory may be related to an unstable Hopf bifurcation, and hence may not exist in theory, even though experiments clearly show that it should); (ii) the transition from one dynamical state to the next; (iii) the amplitude of divergence, i.e., location of the fixed points, and limit-cycle amplitude and frequency for flutter; (iv) the exploration of nonstandard dynamics, such as the quasiperiodic or aperiodic regime between second- and third-mode flutter. To-date, however, all the available theoretical models (Païdoussis 1966*a*, 1973) are linear.

In this paper, a weakly nonlinear equation of motion is derived in a Hamiltonian framework, which in the linear limit is identical to that obtained earlier by Païdoussis (1973). The kinetic and potential energies of the cylinder itself are discussed in Section 3.

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The fluid dynamic forces are introduced in terms of virtual work expressions, separately for the inviscid forces (Section 4.1) and the hydrostatic and frictional forces (Section 4.2), separately for convenience, as discussed in Section 2. The nonlinear equation of motion, as well as the linearized version, are given in Section 5. The boundary conditions are presented in Section 6, while a discussion of the methods of analysis to be used in Part 3 is given in Section 7.

The results of calculations using this theoretical model are presented in Part 3 (Semler *et al.* 2002) of this study, wherein they are compared with the observed behaviour.

2. DEFINITIONS AND PRELIMINARIES

The system consists of a flexible slender cylinder of radius R, length L, flexural rigidity EI and mass per unit length m, centrally located in a channel of radius R_0 and subjected to an axial flow velocity U, as shown in Figure 1. The undeformed cylinder axis coincides with the X-axis, and is in the direction of gravity if the system is not horizontal. The cylinder is cantilevered, generally fitted with an ogival end-piece at the free end (Figure 1), which is considered to be short relative to the overall length of the cylinder. The equations of motion are derived ignoring this noncylindrical segment of the cylinder, which is taken into account with the boundary conditions.

The basic assumptions made for the cylinder and for the fluid are that (i) the fluid is incompressible, (ii) the mean flow velocity is constant, (iii) the cylinder is slender, so that Euler–Bernoulli beam theory is applicable, (iv) although the deflections of the cylinder may be large, strains are small, and (v) the cylinder centre-line is inextensible.

The derivation of the equations of motion is given here with sufficient detail to be followed, but omitting some of the steps in the interests of brevity. The assiduous reader is referred to Lopes *et al.* (1999*a*,*b*) for the full details,† as well as for the derivation of the equation of motion of a cylinder with both ends supported.

Two coordinate systems are used: the Lagrangian (X, Y, Z, T), associated with material points on the undeformed cylinder, and the Eulerian (x, y, z, t), associated with the deformed state of the cylinder. The displacement of a material point is thus, $u = x - X$, $v = y - Y$ and $w = z - Z$ — see Figure 2. For a slender cylinder and motions with the cylinder centre-line in the (X, Y)-plane, we have $Y = 0$ and $z = Z = 0$.

As the cylinder centre-line is assumed to be inextensible,

$$(\partial x/\partial X)^2 + (\partial y/\partial X)^2 = 1. \tag{1}$$

Hence, in this case, one may use the curvilinear coordinate along the cylinder, s, instead of $X : s = X$.‡ One may thus obtain the curvature κ along the deformed cylinder (Semler *et al.* 1994),

$$\kappa = \frac{\partial^2 y/\partial s^2}{\sqrt{1 - (\partial y/\partial s)^2}}. \tag{2}$$

† However, refer to the paragraph following equation (45).

‡ For an extensible cylinder, however, defining ε as the axial strain along the centre-line, X and s are related by $\partial X/\partial s = 1/(1+\varepsilon)$, with $1 + \varepsilon(X) = [(\partial x/\partial X)^2 + (\partial y/\partial X)^2]^{1/2} = [(1 + \partial u/\partial X)^2 + (\partial v/\partial X)^2]^{1/2}$.

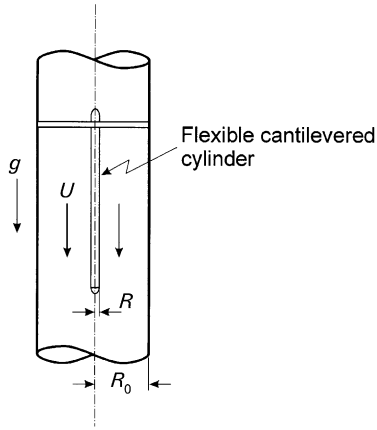

Fig. 1. Diagrammatic view of a vertical cantilevered cylinder in axial flow, in the test-section of a circulating system.

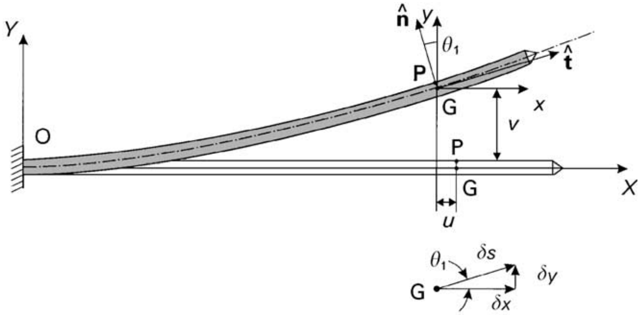

Fig. 2. Diagram defining the coordinate systems. A material point G on the neutral axis of the cylinder at curvilinear coordinate s is located at $G(X, Y)$ before deformation and $G(x, y)$ afterwards; so that its displacement is $\{u, v\} = \{x - X, y - Y\}$. For a point P at a distance y from the centre-line, at the same cross-section, the displacement is $\{u - y\sin\theta_1, v + u(\cos\theta_1 - 1)\}$.

The equation of motion is derived via Hamilton's principle,

$$\delta \int_{t_1}^{t_2} \mathscr{L}\, \mathrm{d}t + \int_{t_1}^{t_2} \delta W\, \mathrm{d}t = 0, \tag{3}$$

where $\mathscr{L} = \mathscr{T}_c - \mathscr{V}_c$ is the Lagrangian, $\mathscr{T}_c$ and $\mathscr{V}_c$ being the kinetic and potential energies of the cylinder, and δW the virtual work by the fluid-related forces acting on the cylinder.

The derivation of the forces due to the fluid proceeds in a similar manner as in Païdoussis (1966*a*, 1973). Thus, the inviscid, viscous and hydrostatic forces are determined separately for convenience — rather than together, say by direct application of the Navier–Stokes equations. The separate derivation of inviscid and viscous forces may be justified by the fact that the former are dominant — considering a large enough Reynolds number. The inviscid forces are derived via slender body potential flow theory, while the viscous forces are formulated semi-empirically. This approach simplifies the analysis considerably and has been shown to give acceptable results (Païdoussis 1966*a*, *b*, 1973, 1979; Païdoussis *et al*. 2002).

In the derivations in Sections 3 and 4, the following relationships have been found to be useful:

$$\delta x = -(y' + \tfrac{1}{2}y'^3)\delta y + \int_0^s (y'' + \tfrac{3}{2}y'^2 y'')\delta y \, \mathrm{d}s + \mathcal{O}(\varepsilon^5), \tag{4a}$$

$$\int_0^L g(s)\left[\int_0^s f(s)\delta y \, \mathrm{d}s\right]\mathrm{d}s = \int_0^L \left[\int_s^L g(s)\,\mathrm{d}s\right] f(s)\delta y \, \mathrm{d}s, \tag{4b}$$

$$\int_s^L (L-s)y'y''\,\mathrm{d}s = -\tfrac{1}{2}(L-s)y'^2 + \int_s^L \tfrac{1}{2}y'^2\,\mathrm{d}s. \tag{4c}$$

The equation of motion is derived correct to third order, $\mathcal{O}(\varepsilon^3)$, for $y = v \sim \mathcal{O}(\varepsilon)$ and, via equation (1), $u \sim \mathcal{O}(\varepsilon^2)$. Hence the expressions for the components of the virtual work δW must be correct to $\mathcal{O}(\varepsilon^3)$, while the energy expressions to $\mathcal{O}(\varepsilon^4)$.

3. KINETIC AND POTENTIAL ENERGIES OF THE CYLINDER

Nonlinear expressions for the kinetic and potential energies of the cylinder itself have been obtained when deriving the nonlinear equations of motion of a pipe conveying fluid. The reader is referred to Semler *et al.* (1994) for details.

The kinetic and potential energies are

$$\mathcal{T}_c = \tfrac{1}{2}m\int_0^L V_c^2\,\mathrm{d}X, \quad \mathcal{V}_c = \tfrac{1}{2}EI\int_0^L \kappa^2\,\mathrm{d}X - mg\int_0^L x\,\mathrm{d}X, \tag{5}$$

where $V_c = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ is the velocity of a cylinder element, and κ is its curvature. The limit L is really $L - l$, where $l \ll L$ is the length of the ogival end, here ignored. After considerable manipulation and use of equations (1) and (4a,b), while keeping in mind the orders of magnitude of the various quantities, we obtain

$$\begin{aligned}\delta\int_{t_1}^{t_2} \mathcal{T}_c\,\mathrm{d}t = &-m\int_{t_1}^{t_2}\int_0^L \Bigg\{\ddot{y} + y'\int_0^s (\dot{y}'^2 + y'\ddot{y}')\,\mathrm{d}s \\ &-y''\int_s^L\int_0^s (\dot{y}'^2 + y'\ddot{y}')\,\mathrm{d}s\,\mathrm{d}s\Bigg\}\delta y\,\mathrm{d}s\,\mathrm{d}t + \mathcal{O}(\epsilon^5),\end{aligned} \tag{6a}$$

$$\begin{aligned}\delta\int_{t_1}^{t_2} \mathcal{V}_c\,\mathrm{d}t = EI&\int_{t_1}^{t_2}\int_0^L [y'''' + 4y'y''y''' + y''^3 + y''''y'^2]\,\delta y\,\mathrm{d}s\,\mathrm{d}t \\ &- mg\int_{t_1}^{t_2}\int_0^L [-(y' + \tfrac{1}{2}y'^3) + (L-s)(y'' + \tfrac{3}{2}y''y'^2)]\,\delta y\,\mathrm{d}s\,\mathrm{d}t \\ &+ \mathcal{O}(\epsilon^5),\end{aligned} \tag{6b}$$

where $()' = \partial()/\partial s$ and $(\dot{\ }) = \partial(\ )/\partial t$.

4. THE FLUID DYNAMIC FORCES

In accordance with the approach adopted, an element of the deformed or oscillating cylinder is subjected to the following set of forces, as shown in Figure 3: the inviscid fluid dynamic force $F_A\delta s$, the normal and longitudinal viscous (frictional) forces, $F_N\delta s$ and

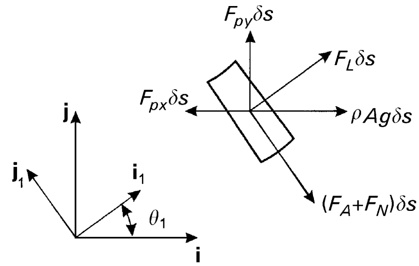

Fig. 3. An element δs of the cylinder, showing the forces acting on it.

$F_L\delta s$, respectively, and the hydrostatic forces $F_{px}\delta s$ and $F_{py}\delta s$ in the x- and y-direction, respectively.†

4.1. The Inviscid Fluid Dynamic Forces

These forces are obtained via slender-body potential flow theory, following closely the Lighthill (1960) formulation. The motion of the body is not represented by the deflection of its centre-line; rather, the motion of every point of the cylinder is taken into account. Hence, the coordinate y is no longer equal to the displacement v, but is of the form $y = Y + v$, where Y describes the position of a point in its original state.

A coordinate transformation is required in order to describe the body in its undeformed state. Hence, it seems appropriate to introduce also Lagrangian coordinates, since the Eulerian coordinate system involves deformation and motion of the body, whereas the Lagrangian system represents the undeformed state of the body. The Eulerian coordinates (x, y, z, t) are related to the Lagrangian ones (X, Y, Z, T) by

$$x(X,T) = X + u(X,T), \quad y(X,T) = Y + v(X,T), \quad z = Z, \quad t = T. \tag{7}$$

We now consider a displacement $y(X, t)$ of the cylinder in the y-direction, away from its stationary, straight configuration. In Figure 4, we introduce the unit vector pair $(\mathbf{i}_1, \mathbf{j}_1)$, respectively in the tangential and normal to the centre-line directions, at angle θ_1 to $(\mathbf{i}, \mathbf{j})$.

We consider next an element of the cylinder as in Figure 4, and define the relative fluid-body velocity, $\mathrm{V} = \dot{\mathbf{y}} + \dot{\mathbf{x}} - \mathbf{U}_f$, in which $\mathbf{U}_f$ is the mean axial flow velocity relative to the deforming cylinder. Then, projecting this on $\mathbf{j}_1$, leads to $V = \dot{y}\cos\theta_1 + (U_f - \dot{x})\sin\theta_1$. From equation (7) we have $\partial x/\partial X = 1 + \partial u/\partial X + \mathcal{O}(\epsilon^4)$, and hence $\theta_1 = \tan^{-1}[(\partial y/\partial X)(1 - \partial u/\partial X)] + \mathcal{O}(\epsilon^5)$. From this, keeping in mind the orders of magnitude of y and u, we obtain

$$\theta_1 = y' - u'y' - \tfrac{1}{3}\, y'^3 + \mathcal{O}(\epsilon^5) \tag{8}$$

and $\cos\theta_1 = 1 - \frac{1}{2}\theta_1^2 + \mathcal{O}(\theta_1^4), \sin\theta_1 = \theta_1 - \frac{1}{6}\theta_1^3 + \mathcal{O}(\theta_1^5)$; therefore,

$$\cos\theta_1 = 1 - \tfrac{1}{2}\, y'^2 + \mathcal{O}(\epsilon^4),\ \sin\theta_1 = y' - u'y' - \tfrac{1}{2}\, y'^3 + \mathcal{O}(\epsilon^5), \tag{9}$$

where‡ it is understood that X has been replaced by s in the derivatives. Hence, returning to V, we can write

$$V(X,t) = \dot{y} + U_f y' - \tfrac{1}{2}\dot{y}y'^2 - U_f u'y' - \tfrac{1}{2}U_f y'^3 - \dot{x}y' + \mathcal{O}(\epsilon^5). \tag{10}$$

†The δs here is an elemental length and should not be confused with variational δx and δy in Sections 3 and 5.
‡For a cylinder with inextensible centreline, this and other expressions are eventually simplified considerably. Thus, since $u' = -\frac{1}{2}y'^2, \sin\theta_1 = y' + \mathcal{O}(\epsilon^5)$.

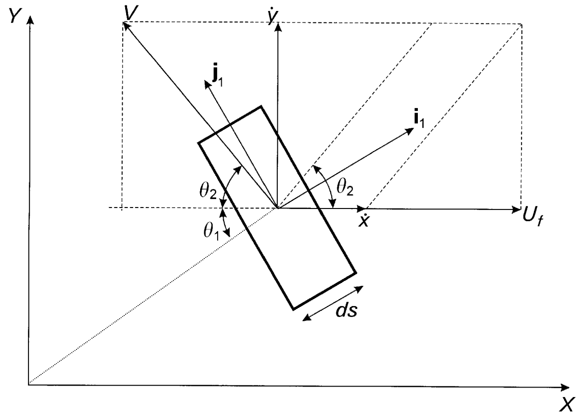

Fig. 4. An element of the cylinder used for the determination of the relative fluid–cylinder velocity V and of the angles θ_1 and θ_2.

Next, the velocity U_f needs to be related to U, the flow velocity relative to the undeformed cylinder. Considering the three-dimensional velocity potential ϕ, we may write $U_f = \partial\phi/\partial x$ and $U = \partial\phi/\partial X$. Then, since $\partial\phi/\partial x = (\partial\phi/\partial X)/(\partial x/\partial X)$, using the expression for $\partial x/\partial X$ obtained previously, we have

$$U_f = U\left(1 - \frac{\partial u}{\partial X}\right) + \mathcal{O}(\epsilon^4). \tag{11}$$

Hence, equation (10) becomes

$$V(X,t) = \dot{y} + Uy' - \tfrac{1}{2}\dot{y}y'^2 - 2Uu'y' - \tfrac{1}{2}Uy'^3 - \dot{x}y' + \mathcal{O}(\epsilon^5). \tag{12}$$

4.1.1. *The pressure distribution*

The velocity potential may be expressed as

$$\phi(X,Y,Z,T) = UX + \phi_0(X,Y,Z) + \phi_1(X,Y,Z,T), \tag{13}$$

where UX is due to the mean flow, ϕ_0 to variations in the body cross-section (here none), and ϕ_1 to motion of the body. This potential must satisfy a number of conditions: (i) the fluid velocity normal to the outer channel is zero; (ii) the fluid does not penetrate the cylinder; (iii) the solution must be 2π-periodic around the cylinder, and even with respect to Z.

The pressure is determined via the Bernoulli equation,

$$P - P_\infty = -\rho\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho(\nabla\phi)^2 + \tfrac{1}{2}\rho U^2. \tag{14}$$

Then, using the relationships for derivatives in the Eulerian and in the Lagrangian coordinates,

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial X} - \frac{\partial u}{\partial X}\frac{\partial}{\partial X} - \frac{\partial v}{\partial X}\frac{\partial}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\frac{\partial}{\partial Y} + \mathcal{O}(\epsilon^5),\\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial Y}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z},\\
\frac{\partial}{\partial t} &= \frac{\partial}{\partial T} - \frac{\partial u}{\partial T}\frac{\partial}{\partial X} - \frac{\partial v}{\partial T}\frac{\partial}{\partial Y} + \frac{\partial u}{\partial T}\frac{\partial v}{\partial X}\frac{\partial}{\partial Y} + \mathcal{O}(\epsilon^5),
\end{aligned} \tag{15}$$

we obtain

$$
\begin{aligned}
P - P_\infty = & -\rho\left(\frac{\partial\phi}{\partial T} - \frac{\partial u}{\partial T}\frac{\partial\phi}{\partial X} - \frac{\partial v}{\partial T}\frac{\partial\phi}{\partial Y} + \frac{\partial u}{\partial T}\frac{\partial v}{\partial X}\frac{\partial\phi}{\partial Y}\right) \\
& -\tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial X} - \frac{\partial u}{\partial X}\frac{\partial\phi}{\partial X} - \frac{\partial v}{\partial X}\frac{\partial\phi}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\frac{\partial\phi}{\partial Y}\right)^2 \\
& -\tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial X}\right)^2 - \tfrac{1}{2}\rho\left(\frac{\partial\phi}{\partial Z}\right)^2 + \tfrac{1}{2}\rho U^2.
\end{aligned} \tag{16}
$$

Substituting now equation (13) in (16), we have

$$
\begin{aligned}
P - P_\infty = & -\rho\left[\frac{\partial\phi_1}{\partial T} - \frac{\partial u}{\partial T}\left(U + \frac{\partial\phi_0}{\partial X} + \frac{\partial\phi_1}{\partial X}\right) - \left(\frac{\partial v}{\partial T} - \frac{\partial u}{\partial T}\frac{\partial v}{\partial X}\right)\left(\frac{\partial\phi_0}{\partial Y} + \frac{\partial\phi_1}{\partial Y}\right)\right] \\
& -\tfrac{1}{2}\rho\left[\left(1 - \frac{\partial u}{\partial X}\right)\left(U + \frac{\partial\phi_0}{\partial X} + \frac{\partial\phi_1}{\partial X}\right) - \left(\frac{\partial v}{\partial X} - \frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\right)\left(\frac{\partial\phi_0}{\partial Y} + \frac{\partial\phi_1}{\partial Y}\right)\right]^2 \\
& -\tfrac{1}{2}\rho\left(\frac{\partial\phi_0}{\partial Y} + \frac{\partial\phi_1}{\partial Y}\right)^2 - \tfrac{1}{2}\rho\left(\frac{\partial\phi_0}{\partial Z} + \frac{\partial\phi_1}{\partial Z}\right)^2 + \tfrac{1}{2}\rho U^2.
\end{aligned} \tag{17}
$$

After many manipulations and truncation to $\mathcal{O}(\epsilon^4)$, we obtain an expression of the form

$$
P - P_\infty = (P - P_\infty)_0 + (P - P_\infty)_2 + (P - P_\infty)_1, \tag{18}
$$

where (i) P_0 is the pressure distribution in steady flow past the undeformed motionless body, (ii) P_2 is the pressure distribution due to steady motion of the cylinder through fluid at rest, and (iii) P_1 is the remainder of the pressure distribution. It is found (Lighthill 1960; Lopes *et al.* 1999*a*, *b*) that

$$
\begin{aligned}
(P - P_\infty)_0 &= P_0 \\
&= -\tfrac{1}{2}\rho\left\{2U\frac{\partial\phi_0}{\partial X} + \left(\frac{\partial\phi_0}{\partial X}\right)^2 + \left(\frac{\partial\phi_0}{\partial Y}\right)^2 + \left(\frac{\partial\phi_0}{\partial Z}\right)^2\right\} + \mathcal{O}(\epsilon^5),
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
(P - P_\infty)_2 = & \; P_2 \\
= & -\rho\left\{-U\frac{\partial u}{\partial T} - \frac{\partial v}{\partial T}\frac{\partial\phi_1}{\partial Y} + \frac{\partial u}{\partial T}\frac{\partial v}{\partial X}\frac{\partial\phi_1}{\partial Y} - U^2\frac{\partial u}{\partial X} + \tfrac{1}{2}U^2\left(\frac{\partial u}{\partial X}\right)^2\right. \\
& + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\frac{\partial\phi_1}{\partial Y} - U\frac{\partial v}{\partial X}\frac{\partial\phi_1}{\partial Y} \\
& \left. + \tfrac{1}{2}\left(\frac{\partial v}{\partial X}\right)^2\left(\frac{\partial\phi_1}{\partial Y}\right)^2 + \tfrac{1}{2}\left(\frac{\partial\phi_1}{\partial Y}\right)^2 + \tfrac{1}{2}\left(\frac{\partial\phi_1}{\partial Z}\right)^2\right\} + \mathcal{O}(\epsilon^5).
\end{aligned} \tag{20}
$$

Neither of the two pressure distributions (19) and (20) contributes to a net force on the body. Net forces result from the unsteady part of the motion of the body and from variations of the cross-section with X, i.e., the pressure distribution $(P - P_\infty)_1 \equiv P_1$. Subtracting equations (20) and (19) from (17) yields the equations for P_1. Furthermore, since ϕ_0 is due to variations of the body cross-section with X, this may further be simplified by taking $\phi_0 = 0$, yielding

$$P_1 = -\rho\left\{\left\{\frac{\partial}{\partial T} + \left[U\left(1 - \frac{\partial u}{\partial X}\right) - \left(\frac{\partial u}{\partial T} + U\frac{\partial u}{\partial X}\right)\right]\frac{\partial}{\partial X}\right\}\phi_1 + \tfrac{1}{2}\left(\frac{\partial \phi_1}{\partial X}\right)^2 - \frac{\partial v}{\partial X}\frac{\partial \phi_1}{\partial Y}\frac{\partial \phi_1}{\partial X}\right\} + \mathcal{O}(\epsilon^5). \tag{21}$$

Once P_1 is determined, the lateral force, or lift, per unit length in the Y-direction may be obtained:

$$\begin{aligned} F_A(X,T) &= \oint_{S_X} P_1(-\mathrm{d}Z) \\ &= -\rho \oint_{S_X} \left\{\left[\frac{\partial}{\partial T} + \left(U\left(1 - \frac{\partial u}{\partial X}\right) - \left(\frac{\partial u}{\partial T} + U\frac{\partial u}{\partial X}\right)\right)\frac{\partial}{\partial X}\right]\phi_1 \right. \\ &\quad \left. + \tfrac{1}{2}\left(\frac{\partial \phi_1}{\partial X}\right)^2 - \frac{\partial v}{\partial X}\frac{\partial \phi_1}{\partial Y}\frac{\partial \phi_1}{\partial X}\right\}(-\mathrm{d}Z) + \mathcal{O}(\epsilon^5), \end{aligned} \tag{22}$$

where the minus sign in front of $\mathrm{d}Z$ enters to reconcile the coordinate axes used here with Lighthill's, and S_X is the circumference of the cylinder.

The next step is to determine ϕ_1, which must satisfy the 2-D Laplace equation, subject to the conditions given below equation (13); e.g., the nonpenetration condition, expressed in cylindrical coordinates, is

$$\begin{aligned} &\frac{\partial \phi_1}{\partial r} + \cos\theta\left(\frac{\partial v}{\partial X}\right)^2\left(\cos\theta\frac{\partial \phi_1}{\partial r} - \sin\theta\frac{\partial \phi_1}{r\partial \theta}\right) + \cos\theta\left(-\frac{\partial v}{\partial T} - U\frac{\partial v}{\partial X}\right. \\ &\left. + \frac{\partial u}{\partial T}\frac{\partial v}{\partial X} + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X} - \frac{\partial v}{\partial X}\frac{\partial \phi_1}{\partial X}\right) + \mathcal{O}(\epsilon^5) = 0 \quad \text{at } r = R, \end{aligned} \tag{23}$$

while on the channel wall we have

$$\left.\frac{\partial \phi_1}{\partial r}\right|_{r=R_0} = 0. \tag{24}$$

4.1.2. *The linear expression for the lift*

It is instructive to solve the linear problem first, as an introduction to the more difficult nonlinear one.

Solutions to the general two-dimensional Laplace equation $\nabla^2 S(r,\theta) = 0$ are of the form

$$\begin{aligned} S(r,\theta) = (A + B\ln r)(C + D\theta) + \sum_{n=0}^{\infty}\{&E_n r^n \cos n\theta + F_n r^n \sin n\theta \\ &+ G_n r^{-n}\cos n\theta + H_n r^{-n}\sin n\theta\}, \end{aligned} \tag{25}$$

where A to D and E_n to H_n are functions of X, subject to the appropriate boundary and regularity conditions.

Simplifying equation (23) to first order, the boundary conditions to be satisfied are

$$\left.\frac{\partial \phi_1}{\partial r}\right|_{r=R_0} = 0, \qquad \left.\frac{\partial \phi_1}{\partial r}\right|_{r=R} = \left(\frac{\partial v}{\partial T} + U\frac{\partial v}{\partial X}\right)\cos\theta,$$
$$\phi_1(r,\theta,X,T) = \phi_1(r,\theta+2\pi,X,T), \quad \phi_1(r,\theta,X,T) = \phi_1(r,-\theta,X,T). \tag{26}$$

According to these boundary conditions and considering the form of the solution for the two-dimensional Laplace equation (25), we can express ϕ_1 in the form

$$\phi_1(r,\theta,X,T) = V(X,T)\Phi(r,\theta,X), \tag{27}$$

where $V(X,T) = [(\partial v/\partial T) + U(\partial v/\partial X)]$ is the linear relative fluid-body velocity. It is noted that Φ is also a solution of the two-dimensional Laplace equation, but with the following boundary conditions:

$$(\partial\Phi/\partial r)\big|_{r=R_0} = 0, \quad (\partial\Phi/\partial r)\big|_{r=R} = \cos\theta, \tag{28a, b}$$

$$\Phi(r,\theta,X) = \Phi(r,\theta+2\pi,X), \quad \Phi(r,\theta,X) = \Phi(r,-\theta,X). \tag{28c, d}$$

The potential Φ is of the form (25), with the coefficients $A, B, \ldots, H_n$ being functions of X. Condition (28d) implies that $D = 0$, and condition (28b) leads to $BC = 0$, $n = 1$, $E_1 - G_1/R^2 = 1$ and $F_1 - H_1/R^2 = 0$. Hence, Φ may be written as

$$\Phi(r,\theta,X) = A' + \left[\left(1 + \frac{G_1}{R^2}\right)r + \frac{G_1}{r}\right]\cos\theta + \left(\frac{H_1}{R^2}r + \frac{H_1}{r}\right)\sin\theta, \tag{29}$$

where $A' = AC$ (in the event that $B = 0$ and $C \neq 0$). Then, applying condition (28a), we obtain $H_1 = 0$ and $G_1 = -R_0^2R^2/(R_0^2 - R^2)$, which leads to

$$\Phi(r,\theta,X) = \left[\left(1 - \frac{R_0^2}{R_0^2 - R^2}\right)r - \frac{R_0^2R^2}{r(R_0^2 - R^2)}\right]\cos\theta, \tag{30}$$

where the constant A' has been suppressed, since the potential can only be determined to within a constant. Notice that condition (28c) is automatically satisfied.†

Next, truncating equation (22) to first order, we obtain the linear expression of the inviscid hydrodynamic force,

$$F_A(X,T) = -\rho\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\oint_{S_X} \phi_1(r,\theta,X,T)(-\mathrm{d}Z). \tag{31}$$

Then, using equations (27) and (30), the integral in equation (31) becomes

$$\oint_{S_X} V(X,T)\Phi(r,\theta,X)(-\mathrm{d}Z) = \int_0^{2\pi} V(X,T)\left(R - 2\frac{R_0^2R}{R_0^2 - R^2}\right)\cos\theta(-R\cos\theta)\,\mathrm{d}\theta$$
$$= \left(\frac{R_0^2 + R^2}{R_0^2 - R^2}\right)V(X,T)\pi R^2 = \chi VA,$$

† For a slender cylinder with $R/L \ll 1$, we have $\Phi(r,\theta,X) = \mathcal{O}(\epsilon)$, and hence, it is verified *a posteriori* that $\phi_1 = V(X,T)\Phi(r,\theta,X)$ is of second order (V being of first order), which is an assumption made in the analysis in conjunction with equation (13).

where χ is the virtual mass coefficient, and A is the cross-sectional area of the cylinder. The inviscid hydrodynamic force is then given by the familiar expression

$$F_A(X,T) = -\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)[MV(X,T)], \tag{32}$$

where $M = \chi\rho A$ is the virtual (added) mass per unit length.

4.1.3. *The lift expression, correct to* $\mathcal{O}(\epsilon^4)$

Let us now consider the nonlinear case. In this case, we define the potential ϕ_1 by

$$\phi_1(r,\theta,X,T) = V(X,T)\Phi(r,\theta,X) + \Psi(r,\theta,X,T), \tag{33}$$

where the potential Ψ is the nonlinear part of ϕ_1, correct to fourth order, which also satisfies a two-dimensional Laplace equation. Again, Ψ needs to be 2π-periodic and even with respect to θ. Concerning the two other boundary conditions, the condition on the outer channel is simply

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=R_0} = 0, \tag{34}$$

whereas the condition of nonpenetration at $r = R$ requires additional manipulations. Substituting equation (33) into (23), and truncating the expression obtained to fourth order, yields

$$\begin{aligned}\left(V(X,T)\frac{\partial\Phi}{\partial r} + \frac{\partial\Psi}{\partial r}\right) &+ \cos\theta\left(\frac{\partial v}{\partial X}\right)^2 V(X,T)\left(\cos\theta\,\frac{\partial\Phi}{\partial r} - \sin\theta\,\frac{\partial\Phi}{r\partial\theta}\right)\\ &+ \cos\theta\left(-V(X,T) + \frac{\partial u}{\partial T}\frac{\partial v}{\partial X} + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X} - \frac{\partial v}{\partial X}\frac{\partial V}{\partial X}\Phi\right)\\ &+ \mathcal{O}(\epsilon^5) = 0,\end{aligned}$$

which, according to condition (28b), can be reduced to

$$\begin{aligned}\left.\frac{\partial\Psi}{\partial r}\right|_{r=R} = &-\left(\frac{\partial u}{\partial T}\frac{\partial v}{\partial X} + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\right)\cos\theta - V\left(\frac{\partial v}{\partial X}\right)^2\left(\cos\theta\,\frac{\partial\Phi}{\partial r} - \sin\theta\,\frac{\partial\Phi}{r\partial\theta}\right)\cos\theta\\ &+ \frac{\partial v}{\partial X}\frac{\partial V}{\partial X}\Phi\cos\theta + \mathcal{O}(\epsilon^5).\end{aligned}$$

Then, replacing Φ by equation (30) at $r = R$, we obtain

$$\begin{aligned}\left.\frac{\partial\Psi}{\partial r}\right|_{r=R} = &-\left(\frac{\partial u}{\partial T}\frac{\partial v}{\partial X} + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\right)\cos\theta - \left(a - \frac{b}{2}\right)V\left(\frac{\partial v}{\partial X}\right)^2\cos\theta\\ &+ \frac{b}{2}V\left(\frac{\partial v}{\partial X}\right)^2\cos 3\theta + \frac{\partial v}{\partial X}\frac{\partial V}{\partial X}(a+b)\,\frac{R}{2}(1+\cos 2\theta) + \mathcal{O}(\epsilon^5),\end{aligned} \tag{35}$$

where $a = -R^2/(R_0^2 - R^2)$ and $b = -R_0^2/(R_0^2 - R^2)$. Consequently, considering conditions (34), (35) and the form of the solutions (25), we finally obtain

$$\begin{aligned}\Psi(r,\theta,X) = &\left[-\left(\frac{\partial u}{\partial T}\frac{\partial v}{\partial X} + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\right) - \left(a - \frac{b}{2}\right)V\left(\frac{\partial v}{\partial X}\right)^2\right]\left(ar + b\frac{R^2}{r}\right)\cos\theta \\ &+ \frac{b}{6}V\left(\frac{\partial v}{\partial X}\right)^2\left[\left(-\frac{R^4}{R_0^6 - R^6}\right)r^3 - \frac{R^4}{r^3}\frac{R_0^6}{R_0^6 - R^6}\right]\cos 3\theta \\ &+ \frac{\partial v}{\partial X}\frac{\partial V}{\partial X}(a+b)\frac{R}{4}\left[\left(-\frac{R^3}{R_0^4 - R^4}\right)r^2 - \frac{R^3}{r^2}\frac{R_0^4}{R_0^4 - R^4}\right]\cos 2\theta \\ &+ \frac{\partial v}{\partial X}\frac{\partial V}{\partial X}(a+b)\frac{R^2}{2}\ln r + \mathcal{O}(\epsilon^5),\end{aligned} \tag{36}$$

where the last term satisfies condition (34) only for R_0 large, i.e., $R/R_0 \ll 1$. Actually, this is not an excessive requirement: one can verify that a ratio of 0·1 for R/R_0, which is indeed relatively large compared to $R/L \sim 0{\cdot}025$ for a slender body, leads to $[(a+b)R^2/2R_0] \sim 0{\cdot}05R$ (the coefficient of the derivative of the last term with respect to r at $r = R_0$). Hence, apart from special cases where the outer channel is very close to the cylinder, we may conclude that this approximation is reasonable.

Substituting now the expression for ϕ_1 into equation (22) and truncating at fourth order yields

$$\begin{aligned}F_A(X,T) = -\rho \oint_{S_X} \Bigg\{ &\left[\frac{\partial}{\partial T} + \left(U\left(1 - \frac{\partial u}{\partial X}\right) - \left(\frac{\partial u}{\partial T} + U\frac{\partial u}{\partial X}\right)\right)\frac{\partial}{\partial X}\right]V\Phi \\ &+ \left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\Psi + \tfrac{1}{2}\left(\frac{\partial V}{\partial X}\right)^2\Phi^2 \\ &- V\frac{\partial v}{\partial X}\frac{\partial V}{\partial X}\frac{\partial \Phi}{\partial Y}\Phi\Bigg\}(-\mathrm{d}Z) + \mathcal{O}(\epsilon^5).\end{aligned} \tag{37}$$

Then, substituting Φ by equation (30) and Ψ by equation (36) into (33), with special attention to the integration of products of cosines, the inviscid hydrodynamic force (37) becomes

$$\begin{aligned}F_A(X,T) = -\rho A\Bigg\{&\left[\left(\frac{\partial}{\partial T} + \left(U\left(1 - \frac{\partial u}{\partial X}\right) - \left(\frac{\partial u}{\partial T} + U\frac{\partial u}{\partial X}\right)\right)\frac{\partial}{\partial X}\right)\right. \\ &\left.\times\left(\chi V - \chi\left(\frac{\partial u}{\partial T}\frac{\partial v}{\partial X} + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\right) - \bar{\chi}V\left(\frac{\partial v}{\partial X}\right)^2\right)\right] - \bar{\chi}V\frac{\partial v}{\partial X}\frac{\partial V}{\partial X}\Bigg\} + \mathcal{O}(\epsilon^5),\end{aligned} \tag{38}$$

where

$$\chi = -(a+b) = \frac{R_0^2 + R^2}{R_0^2 - R^2} \quad\text{and}\quad \bar{\chi} = -(a+b)(a - \tfrac{1}{2}b) = \tfrac{1}{2}\frac{(R_0^2 + R^2)(R_0^2 - 2R^2)}{(R_0^2 - R^2)^2} \tag{39}$$

are virtual mass coefficients.

We notice that, as R_0 becomes large, $\chi \to 1$ and $\bar{\chi} \to \frac{1}{2}$, and in that case

$$\begin{aligned}F_A(X,T) = -&\left\{\frac{\partial}{\partial T} + \left[U\left(1 - \frac{\partial u}{\partial X}\right) - \left(\frac{\partial u}{\partial T} + U\frac{\partial u}{\partial X}\right)\right]\frac{\partial}{\partial X}\right\} \\ &\times\left[V - \left(\frac{\partial u}{\partial T}\frac{\partial v}{\partial X} + 2U\frac{\partial u}{\partial X}\frac{\partial v}{\partial X}\right) - \tfrac{1}{2}V\left(\frac{\partial v}{\partial X}\right)^2\right]M + \tfrac{1}{2}MV\frac{\partial v}{\partial X}\frac{\partial V}{\partial X} + \mathcal{O}(\epsilon^5),\end{aligned} \tag{40}$$

where $V = (\partial v/\partial T) + U(\partial v/\partial X)$, and $M = \rho A$. For the purposes of this analysis, equation (40) is considered to be valid even for cases where R_0 is not so large, simply replacing $M = \rho A$ by $M = \chi\rho A$ in the equation.

4.2. Hydrostatic and Frictional Forces

The hydrostatic forces F_{px} and F_{py}, the resultants of the steady-state pressure p acting on the cylinder, are derived by the procedure in Païdoussis (1973).

Consider an element δs of the cylinder, momentarily frozen and immersed in fluid *on all sides*. Hence, in addition to $F_{px}\delta s$ and $F_{py}\delta s$, the resultants on the normally wet surfaces, there are additional forces, pA and $pA + [\partial(pA)/\partial s]\delta s$ on the two flat, normally dry, surfaces of the element. The net resultant of all these forces is known: it is the buoyancy force. The pressure is assumed to be of the form $p(x) = a + bx$ — which covers both purely hydrostatic and pressure-drop-modified pressure distributions. Consequently, one may write

$$\begin{aligned}&\left[-F_{px} - \frac{\partial}{\partial s}(pA\cos\theta_1)\right]\delta s\,\mathbf{i} + \left[F_{py} - \frac{\partial}{\partial s}(pA\sin\theta_1)\right]\delta s\,\mathbf{j}\\ &= \oiint p\mathbf{n}\,\mathrm{d}A = -\iiint \boldsymbol{\nabla} p\,\mathrm{d}(\mathrm{vol}) = \left(-\frac{\partial p}{\partial x}\,\mathbf{i}\right)A\delta s,\end{aligned} \tag{41}$$

where $\mathbf{n}$ is the outwards pointing normal; to evaluate the right-hand side, use has been made of the fact that the elemental volume is $A\delta s$ and that the pressure gradient is in the x-direction only. Then, using the relations linking $\partial/\partial x, \partial/\partial X$ and $\partial/\partial s$, the fact that $\partial A/\partial X = 0$, and also referring to the inset diagram in Figure 2, one obtains†

$$\begin{aligned}-F_{px} &= \frac{\partial p}{\partial x}A\cos^2\theta_1 + pA\frac{\partial}{\partial X}(\cos\theta_1) - \frac{\partial p}{\partial x}A + \mathcal{O}(\epsilon^4),\\ F_{py} &= \frac{\partial p}{\partial x}A\cos\theta_1\sin\theta_1 + pA\frac{\partial}{\partial X}(\sin\theta_1) + \mathcal{O}(\epsilon^5).\end{aligned}$$

Here, $\partial p/\partial x$ has been used in preference to $\partial p/\partial X$ since $p = p(x)$. Next, using expressions (9), we obtain

$$\begin{aligned}-F_{px} &= -y'^2A(\partial p/\partial x) - y'y''Ap + \mathcal{O}(\epsilon^4),\\ F_{py} &= (y' - u'y' - y'^3)A(\partial p/\partial x) + (y'' - u''y' - u'y'' - \tfrac{3}{2}y'^2y'')Ap + \mathcal{O}(\epsilon^5);\end{aligned} \tag{42}$$

it is recalled that $(\,)' = \partial(\,)/\partial s \equiv \partial(\,)/\partial X \neq \partial(\,)/\partial x$.

Furthermore, by assuming the lateral movement of the cylinder to have a negligible effect on the axial pressure distribution in the fluid at large, one can write (Païdoussis 1973)

$$A\left(\frac{\partial p}{\partial x}\right) = -\frac{1}{2}\rho DU^2\,C_T\frac{D}{D_h} + \rho gA, \tag{43}$$

where D_h is the hydraulic diameter, and C_T is a friction coefficient — cf. equations (46). Rewriting the derivative with respect to X by using one of the relationships of the last

† The following relationships are recalled: (i) $x = X + u$, thus $\partial x/\partial X = 1 + \partial u/\partial X$; (ii) $\partial X/\partial s = 1/(1+\varepsilon)$, ε being the centre-line extension; here $(1+\varepsilon) = [(1+\partial u/\partial X)^2 + (\partial v/\partial X)^2]^{1/2}$; (iii) $\partial x/\partial s = (\partial x/\partial X)(\partial X/\partial s) = (1+u')/(1+\varepsilon) = \cos\theta_1$. For an inextensible centre-line, $\varepsilon = 0, \partial X/\partial s = 1, \partial x/\partial X \simeq 1 - \frac{1}{2}y'^2$. Hence, for example, $\partial(pA\cos\theta_1)/\partial s = (\partial(pA)/\partial s)\cos\theta_1 + pA(\partial X/\partial s)(\partial\cos\theta_1/\partial X) = (\partial p/\partial x)A\cos^2\theta_1 + pA(\partial\cos\theta_1/\partial X)$.

footnote and integrating from $X = s$ to L, we obtain

$$Ap(s) = Ap(L) + \left(\frac{1}{2}\rho D U^2 C_T \frac{D}{D_h} - \rho g A\right)\left[(L-s) - \int_s^L \tfrac{1}{2} y'^2\,\mathrm{d}s\right] + \mathcal{O}(\epsilon^4), \tag{44}$$

an expression used in the last steps of the formulation of the equation of motion.

Finally, introducing these expressions into equations (42) we obtain

$$\begin{aligned} -F_{px} &= y'^2\left(\tfrac{1}{2}\rho D U^2\, C_T \frac{D}{D_h} - \rho g A\right) - y' y'' Ap + \mathcal{O}(\epsilon^4),\\ F_{py} &= (y' - u'y' - y'^3)\left(-\tfrac{1}{2}\rho D U^2 C_T \frac{D}{D_h} + \rho g A\right)\\ &\quad + (y'' - u''y' - u'y'' - \tfrac{3}{2} y'^2 y'')Ap + \mathcal{O}(\epsilon^5). \end{aligned} \tag{45}$$

Here it is noted that in the derivation in Lopes *et al.* (1999*a*), the expressions for F_{px} and F_{py} are slightly different, having been obtained with the simplifying assumption that $\partial p/\partial x = \partial p/\partial X$, which is correct to first order, but strictly incorrect to third order. The resulting expressions are given in Appendix A since the equations incorporating these expressions have been used in the calculations in Part 3.

Next, we proceed with the formulation of the viscous, frictional forces, on the basis of the semi-empirical expressions proposed by Taylor (1952). These are considered to be adequate, unless confinement of the flow by the channel is very severe — in which case, they could be obtained from the unsteady pressures derived analytically by Mateescu *et al.* (1994*a*, *b*), for instance, but unfortunately not in closed form. An alternative is to use the semi-empirical data, but which again are not available in easily usable closed form, compiled in Païdoussis (1998, 2002); however, these can be fitted in the framework of the Taylor formulation, and hence we proceed with that.

The expressions proposed by Taylor are

$$F_N = \tfrac{1}{2}\rho D U^2 (C_N \sin i + C_{Dp} \sin^2 i), \quad F_L = \tfrac{1}{2}\rho D U^2 C_T \cos i, \tag{46}$$

where C_N and C_T are friction coefficients and C_{Dp} a form-drag coefficient; i is the angle of attack. In some of the analysis, distinct C_N and C_T are used; frequently, however, the simplified form $C_N = C_T = C_f$ is taken instead. F_N and F_L act in the $-\mathbf{j}_1$ and $\mathbf{i}_1$ direction, respectively. Expressing $i = \theta_1 + \theta_2$, we note that we have already expressions for θ_1 in equations (8) and (9). Proceeding similarly, we have $\theta_2 = \tan^{-1}\{(\partial y/\partial t)/[U_f - (\partial x/\partial t)]\}$ — see Figure 4 — and hence

$$\theta_2 = \frac{\dot{y}}{U_f} + \frac{\dot{x}\dot{y}}{U_f^2} - \frac{1}{3}\frac{\dot{y}^3}{U_f^3} + \mathcal{O}(\epsilon^5). \tag{47}$$

We can, therefore, find i, cos i and sin i, as follows:

$$\begin{aligned} i &= y' + \frac{\dot{y}}{U_f} - u'y' + \frac{\dot{x}\dot{y}}{U_f^2} - \frac{1}{3}\left(y'^3 + \frac{\dot{y}^3}{U_f^3}\right) + \mathcal{O}(\epsilon^5),\\ \cos i &= 1 - \frac{1}{2}\left(y'^2 + 2\frac{y'\dot{y}}{U_f} + \frac{\dot{y}^2}{U_f^2}\right) + \mathcal{O}(\epsilon^4),\\ \sin i &= y' + \frac{\dot{y}}{U_f} - u'y' + \frac{\dot{x}\dot{y}}{U_f^2} - \frac{1}{2}\left(y'^3 + \frac{\dot{y}^3}{U_f^3} + \frac{y'^2\dot{y}}{U_f} + \frac{y'\dot{y}^2}{U_f^2}\right) + \mathcal{O}(\epsilon^5). \end{aligned} \tag{48}$$

Substituting these in equation (46), and relating U_f to U through equation (11), we obtain

$$F_N = \tfrac{1}{2}\rho D U^2 \left[C_N \left(y' + \frac{\dot y}{U} + \frac{\dot y u'}{U} - u'y' + \frac{\dot x \dot y}{U^2} - \frac{1}{2}\left(y'^3 + \frac{\dot y^3}{U^3} + \frac{y'^2 \dot y}{U} + \frac{y' \dot y^2}{U^2} \right) \right) + C_{Dp}\left(y'^2 + 2\frac{y'\dot y}{U} + \frac{\dot y^2}{U^2} \right) \right] + \mathcal{O}(\epsilon^5),$$
$$F_L = \tfrac{1}{2}\rho D U^2 C_T \left[1 - \frac{1}{2}\left(y'^2 + 2\frac{y'\dot y}{U} + \frac{\dot y^2}{U^2} \right) \right] + \mathcal{O}(\epsilon^4). \quad (49)$$

The quadratic terms in the expression for F_N need to be modified in order to obtain forces which are odd with respect to y' and $\dot y$, thus forces always opposing motion. Triantafyllou & Chryssostomidis (1989) have shown that terms of the form $[y' + (\dot y/U)]^2$, which are found in the normal viscous forces, could be written as $[y' + (\dot y/U)]|(y' + (\dot y/U))|$ directly. In the same spirit, expressing y'^2 as $y'|y'|$, $\dot y^2$ as $\dot y|\dot y|$, and so on in F_N, we obtain

$$F_N = \tfrac{1}{2}\rho D U^2 \left[C_N \left(y' + \frac{\dot y}{U} + \frac{\dot y u'}{U} - u'y' + \frac{\dot x \dot y}{U^2} - \frac{1}{2}\left(y'^3 + \frac{\dot y^3}{U^3} + \frac{y'^2 \dot y}{U} + \frac{y' \dot y^2}{U^2} \right) \right) + C_{Dp}\left(y'|y'| + \frac{y'|\dot y| + |y'|\dot y}{U} + \frac{\dot y|\dot y|}{U^2} \right) \right] + \mathcal{O}(\epsilon^5). \quad (50)$$

On the other hand, the longitudinal force is even with respect to y' to $\dot y$, and hence no such modification is necessary.

5. THE EQUATION OF MOTION

The virtual work associated with the fluid-dynamic forces may be expressed as

$$\int_{t_1}^{t_2} \delta W \, \mathrm{d}t = \int_{t_1}^{t_2} \int_0^L \{[-F_{px} + F_L \cos\theta_1 + (F_A + F_N)\sin\theta_1]\delta x + [F_{py} + F_L \sin\theta_1 - (F_A + F_N)\cos\theta_1]\delta y\} \, \mathrm{d}s \, \mathrm{d}t. \quad (51)$$

Substituting in the above equation the expressions for F_A, F_{px}, etc. derived in Section 4, and utilizing equations (4a–c) and (6a,b), one obtains with the aid of equation (3), the nonlinear equation of motion:

$$(m+M)\ddot y + 2MU\dot y'(1 + \tfrac{7}{4}y'^2) + MU^2 y''(1 + \tfrac{5}{2}y'^2) - \tfrac{3}{2}M\dot y y'(\dot y' + Uy'')$$
$$+ \tfrac{1}{2}\rho D U^2 C_N (y' + \tfrac{1}{2}y'^3) - \tfrac{1}{2}\rho D U^2 C_T (L-s)(y'' + \tfrac{3}{2}y'^2 y'') - Ap(L)(y'' + y'^2 y'')$$
$$+ (\tfrac{1}{2}\rho D U^2 C_T h + mg - \rho g A)[y' + \tfrac{1}{2}y'^3 - (L-s)(y'' + \tfrac{3}{2}y'^2 y'')]$$
$$+ EI(y'''' + 4y'y''y''' + y''^3 + y''''y'^2) - \tfrac{1}{2}\rho D C_N \dot y \int_0^s y'\dot y' \, \mathrm{d}s$$
$$+ \tfrac{1}{2}\rho D U^2 C_N \left(\frac{\dot y}{U} - \frac{1}{2}\frac{y'\dot y^2}{U^2} - \frac{1}{2}\frac{y'^2 \dot y}{U} - \frac{1}{2}\frac{\dot y^3}{U^3} \right) + \tfrac{1}{2}\rho D U^2 C_{Dp}\left(y'|y'| + \frac{y'|\dot y| + \dot y|y'|}{U} + \frac{\dot y|\dot y|}{U^2} \right)$$
$$- my'' \int_s^L \int_0^s (\dot y'^2 + y'\ddot y') \, \mathrm{d}s \, \mathrm{d}s + 2M(\dot y' + Uy'') \int_0^s y'\dot y' \, \mathrm{d}s$$

$$
\begin{aligned}
&- My'' \int_s^L (\ddot{y}y' + 2U\dot{y}'y' + U^2 y''y')\,\mathrm{d}s + (m+M)y' \int_0^s (\dot{y}'^2 + y'\ddot{y}')\,\mathrm{d}s \\
&+ y'' \int_s^L \{Ap(L)y'y'' + \tfrac{1}{4}\rho D C_T \dot{y}^2\}\,\mathrm{d}s \\
&+ \tfrac{1}{2}\rho D U^2 y''(C_T - C_N) \int_s^L \left(y'^2 + \frac{y'\dot{y}}{U} \right) \mathrm{d}s + \mathcal{O}(\epsilon^5) = 0. \qquad (52)
\end{aligned}
$$

In this equation, it is realized that, unless there is a drogue at the free end, $p(L)$ would normally arise from base drag at the free end of the cylinder, in which case $Ap(L)$ may be expressed as $\frac{1}{2}\rho D^2 U^2 C_b$, where C_b is the base drag coefficient. Defining next the dimensionless quantities

$$
\begin{aligned}
&\xi = \frac{s}{L}, \quad \eta = \frac{y}{L}, \quad \tau = \left(\frac{EI}{m+\rho A} \right)^{1/2} \frac{t}{L^2}, \quad \mathcal{U} = \left(\frac{\rho A}{EI} \right)^{1/2} UL, \\
&\beta = \frac{\rho A}{m+\rho A}, \quad \gamma = \frac{(m-\rho A)gL^3}{EI}, \quad c_N = \frac{4}{\pi} C_N, \quad c_T = \frac{4}{\pi} C_T, \qquad (53) \\
&c_d = \frac{4}{\pi} C_{Dp}, \quad \varepsilon = \frac{L}{D}, \quad h = \frac{D}{D_h}, \quad c_b = \frac{4}{\pi} C_b,
\end{aligned}
$$

one obtains the dimensionless equation of motion:

$$
\begin{aligned}
&[1 + (\chi - 1)\beta]\ddot{\eta} + 2\mathcal{U}\sqrt{\beta}\chi\dot{\eta}'(1 + \tfrac{7}{4}\eta'^2) + \mathcal{U}^2\chi\eta''(1 + \tfrac{5}{2}\eta'^2) - \tfrac{3}{2}\chi\dot{\eta}\eta'(\beta\dot{\eta}' + \mathcal{U}\sqrt{\beta}\eta'') \\
&+ \tfrac{1}{2}\mathcal{U}^2\varepsilon c_N[\eta' + \tfrac{1}{2}\eta'^3] - \tfrac{1}{2}\mathcal{U}^2\varepsilon c_T(1-\xi)(\eta'' + \tfrac{3}{2}\eta'^2\eta'') - \tfrac{1}{2}c_b\mathcal{U}^2(\eta'' + \eta'^2\eta'') \\
&+ (\tfrac{1}{2}\mathcal{U}^2\varepsilon c_T h + \gamma)[\eta' + \tfrac{1}{2}\eta'^3 - (1-\xi)(\eta'' + \tfrac{3}{2}\eta'^2\eta'')] \\
&+ \eta'''' + 4\eta'\eta''\eta''' + \eta''^3 + \eta''''\eta'^2 - \tfrac{1}{2}\varepsilon c_N\beta\dot{\eta}\int_0^\xi \eta'\dot{\eta}'\,\mathrm{d}\xi \\
&+ \tfrac{1}{2}\mathcal{U}^2\varepsilon c_N\left(\frac{\sqrt{\beta}}{\mathcal{U}}\dot{\eta} - \frac{1}{2}\frac{\beta}{\mathcal{U}^2}\dot{\eta}^2\eta' - \frac{1}{2}\frac{\sqrt{\beta}}{\mathcal{U}}\dot{\eta}\eta'^2 - \frac{1}{2}\frac{\beta^{3/2}}{\mathcal{U}^3}\dot{\eta}^3 \right) \\
&+ \tfrac{1}{2}\mathcal{U}^2\varepsilon c_d\left(\eta'|\eta'| + \frac{\sqrt{\beta}}{\mathcal{U}}(\dot{\eta}|\eta'| + \eta'|\dot{\eta}|) + \frac{\beta}{\mathcal{U}^2}\dot{\eta}|\dot{\eta}| \right) \\
&- \eta''(1-\beta)\int_\xi^1 \int_0^\xi (\dot{\eta}'^2 + \eta'\ddot{\eta}')\,\mathrm{d}\xi\,\mathrm{d}\xi + 2\chi(\beta\dot{\eta}' + \mathcal{U}\sqrt{\beta}\eta'')\int_0^\xi \eta'\dot{\eta}'\,\mathrm{d}\xi \\
&- \chi\eta''\int_\xi^1 (\beta\ddot{\eta}\eta' + 2\mathcal{U}\sqrt{\beta}\dot{\eta}'\eta' + \mathcal{U}^2\eta''\eta')\,\mathrm{d}\xi + \eta'(1 + (\chi-1)\beta)\int_0^\xi (\dot{\eta}'^2 + \eta'\ddot{\eta}')\,\mathrm{d}\xi \\
&+ \eta''\int_\xi^1 \{\tfrac{1}{2}c_b\mathcal{U}^2\eta'\eta'' + \tfrac{1}{4}\varepsilon c_T\beta\dot{\eta}^2\}\,\mathrm{d}\xi \\
&+ \tfrac{1}{2}\mathcal{U}^2\eta''(\varepsilon c_T - \varepsilon c_N)\int_\xi^1 \left(\eta'^2 + \frac{\sqrt{\beta}}{\mathcal{U}}\eta'\dot{\eta} \right)\mathrm{d}\xi + \mathcal{O}(\epsilon^5) = 0, \qquad (54)
\end{aligned}
$$

where $(\ )' = \partial(\ )/\partial\xi, (\dot{\ }) = \partial(\ )/\partial\tau$.

Since some linear calculations are performed in Parts 1 and 3 of this work (Païdoussis *et al.* 2002; Semler *et al.* 2002), the linearized version of equation (54) is given below, for completeness:

$$[1+(\chi-1)\beta]\ddot{\eta}+2\mathscr{U}\sqrt{\beta}\chi\dot{\eta}'+\mathscr{U}^2\chi\eta''-[\gamma+\tfrac{1}{2}\mathscr{U}^2\varepsilon c_T(1+h)](1-\xi)\eta'' \\ +[\tfrac{1}{2}\mathscr{U}^2(\varepsilon c_N+\varepsilon c_T h)+\gamma]\eta'+\eta''''+\tfrac{1}{2}\mathscr{U}\varepsilon c_N\sqrt{\beta}\dot{\eta}-\tfrac{1}{2}c_b\mathscr{U}^2\eta''=0. \quad (55)$$

This equation is identical to that given by Païdoussis (1973) if (i) dissipation in the material of the cylinder is neglected, (ii) a term equal to $+\frac{1}{2}\varepsilon c_d\sqrt{\beta}\dot{\eta}$ is added, representing an arbitrary (nonmathematical) linearization of the damping in stagnant fluid, and (iii) $c_N = c_T = c_f$ is taken. The dissipation can be taken into account by replacing E by $\{E^* \times (\partial/\partial t) + E\}$ in the dimensional version of the equations of motion. However, flow-induced damping is much more important for cantilevered cylinders, an inherently nonconservative system, and hence this term would not change the qualitative dynamics of the system (except at $\mathscr{U} = 0$), nor sensibly the quantitative dynamics.

6. BOUNDARY CONDITIONS

It is supposed that at its free end the cylinder is terminated by a short, ogival end, the cross-sectional area of which varies smoothly from A to zero in a distance $l \ll L$. Further, this ogival end is assumed to be rigid, so that its motion is determined solely by the values of displacement and velocity at $s = L - l$; y, y', V and δy are all constant with s. The boundary conditions are derived to first order, i.e. correct to $\mathcal{O}(\epsilon)$.

The variation of the Lagrangian of the ogival end is

$$\delta\int_{t_1}^{t_2}\mathscr{L}\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_{L-l}^{L}[\rho_c A(s)(\ddot{x}\delta x+\ddot{y}\delta y)-\rho_c A(s)g\delta x]\,\mathrm{d}s\,\mathrm{d}t. \quad (56)$$

It is convenient to re-write this in terms of virtual displacements in the longitudinal and transverse directions, δu_L and δu_N, respectively,

$$\begin{Bmatrix}\delta x\\ \delta y\end{Bmatrix} = \begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\begin{Bmatrix}\delta u_L\\ \delta u_N\end{Bmatrix}, \quad (57)$$

leading to

$$\delta\int_{t_1}^{t_2}\mathscr{L}\,\mathrm{d}t = -\int_{t_1}^{t_2} m[\ddot{y}(\sin\theta_1\,\delta u_L+\cos\theta_1\,\delta u_N)+\ddot{x}(\cos\theta_1\,\delta u_L-\sin\theta_1\,\delta u_N)]s_\mathrm{e}\,\mathrm{d}t \\ +\int_{t_1}^{t_2} mgs_\mathrm{e}(\cos\theta_1\,\delta u_L-\sin\theta_1\,\delta u_N)\,\mathrm{d}t+\mathcal{O}(\epsilon^3), \quad (58)$$

in which $s_\mathrm{e} = (1/A)\int_{L-l}^{L} A(s)\,\mathrm{d}s$. Furthermore, since $\ddot{x}$ is of second order, equation (58) reduces to

$$\delta\int_{t_1}^{t_2}\mathscr{L}\,\mathrm{d}t = -\int_{t_1}^{t_2}[m\ddot{y}s_\mathrm{e}\delta u_N - mgs_\mathrm{e}(\delta u_L - y'\delta u_N)]\,\mathrm{d}t+\mathcal{O}(\epsilon^3). \quad (59)$$

Considering next the virtual work by all the fluid-dynamic forces acting on the tapering end (Figure 5), and following equation (51), we have

$$\int_{t_1}^{t_2}\delta\mathscr{W}\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{L-l}^{L}\{[-F_{px}+F_L\cos\theta_1+(F_A+F_N)\sin\theta_1]\delta x \\ +[F_{py}+F_L\sin\theta_1-(F_A+F_N)\cos\theta_1]\delta y\}\,\mathrm{d}s\,\mathrm{d}t, \quad (60)$$

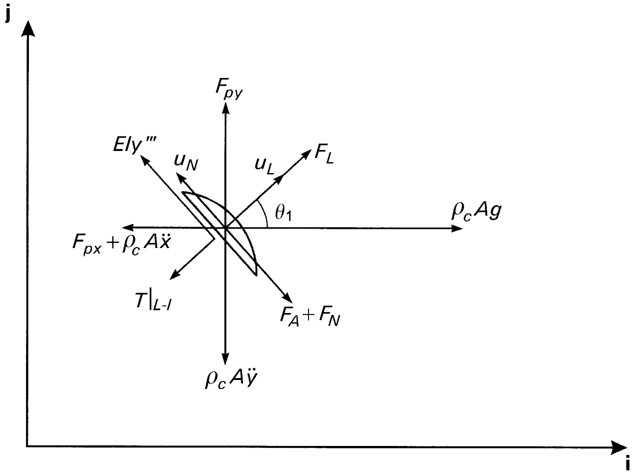

Fig. 5. The end-piece at the free end of the cylinder showing the forces acting on it; ρ_c is the density of the cylinder and of the end-piece. The notation in this figure has been simplified; most of the "forces" shown, e.g. F_L, are really forces per unit length and should be understood to stand for $\int_{L-l}^{L} F_L \,\mathrm{d}s$, etc.

which, in view of equation (57), may be written as

$$\int_{t_1}^{t_2} \delta\mathscr{W}\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{L-l}^{L} \{[-F_{px}\cos\theta_1 + F_{py}\sin\theta_1 + F_L]\delta u_L + [F_{px}\sin\theta_1 + F_{py}\cos\theta_1 - (F_A+F_N)\delta u_N\}\,\mathrm{d}s\,\mathrm{d}t. \tag{61}$$

In this expression, F_N and F_L are not constant, since the diameter is a function of s, and similarly F_{px}, F_{py} and F_A involve the cross-sectional area, also varying with s. Thus, noting that θ_1, y' and δy are constant for $L-l<s<L$ and that $y''=0$, we can write

$$\begin{aligned}\int_{t_1}^{t_2} \delta\mathscr{W}\,\mathrm{d}t = \int_{t_1}^{t_2} \Bigg\{ &-\cos\theta_1 \int_{L-l}^{L} F_{px}\delta u_L\,\mathrm{d}s \\ &+ \sin\theta_1 \int_{L-l}^{L} F_{py}\delta u_L\,\mathrm{d}s + \int_{L-l}^{L} F_L\delta u_L\,\mathrm{d}s \\ &+ \sin\theta_1 \int_{L-l}^{L} F_{px}\delta u_N\,\mathrm{d}s + \cos\theta_1 \int_{L-l}^{L} F_{py}\delta u_N\,\mathrm{d}s \\ &- f\int_{L-l}^{L} F_A\delta u_N\,\mathrm{d}s - \int_{L-l}^{L} F_N\delta u_N\,\mathrm{d}s \Bigg\}\,\mathrm{d}t,\end{aligned} \tag{62}$$

in which the parameter f $(0 \leqslant f \leqslant 1)$ has been introduced in the terms involving F_A, since the ideal inviscid hydrodynamic force will generally not materialize fully over the ogival end because (i) the lateral flow is not truly two-dimensional, some fluid going axially "around" rather than transversely over the tapering end, and (ii) over part of the ogival end, there is boundary layer separation (Païdoussis 1966*a*). Thus, $f=1$ is the ideally slender case, impossible in practice, while normally $0 \leqslant f < 1$. In Parts 2 and 3 of this work, $f \to 1$ is taken for a well-streamlined end, while $f \to 0$ for a blunt end; see also last paragraph of this section.

In equation (62) the simplified expressions for the various forces—leading to a linear final boundary condition—are

$$
\begin{aligned}
-F_{px} &= p(\mathrm{d}A(s)/\mathrm{d}s) + \mathcal{O}(\epsilon^2), \quad F_{py} = y'\left(-\tfrac{1}{2}\rho U^2 C_T \frac{D^2(s)}{D_h} + \rho g A(s) + p(\mathrm{d}A(s)/\mathrm{d}s)\right) + \mathcal{O}(\epsilon^3),\\
F_N &= \tfrac{1}{2}\rho D(s) U C_N(\dot{y} + Uy') + \mathcal{O}(\epsilon^2), \quad F_L = \tfrac{1}{2}\rho D(s) U^2 C_T + \mathcal{O}(\epsilon^2), \qquad (63)\\
F_A &= \chi\rho(\ddot{y} + U\dot{y}')A(s) + \chi\rho U(\dot{y} + Uy')(\mathrm{d}A(s)/\mathrm{d}s) + \mathcal{O}(\epsilon^3),\\
&\cos\theta_1 = 1 + \mathcal{O}(\epsilon^2), \quad \sin\theta_1 = y' + \mathcal{O}(\epsilon^3),
\end{aligned}
$$

having also made use of equations (43) and (44).

Substituting equation (63) into (62), and combining with equation (59) as in (3), after several manipulations and simplifications one obtains

$$
\begin{aligned}
&\delta\int_{t_1}^{t_2}\mathcal{L}\,\mathrm{d}t + \int_{t_1}^{t_2}\delta\mathcal{W}\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\{(-(pA)|_{L-l} + \tfrac{1}{2}\rho DU^2 C_T h s_e + \tfrac{1}{2}\rho D U^2 C_T \bar{s}_e + (m-\rho A)g s_e)\}\delta u_L\,\mathrm{d}t\\
&+ \int_{t_1}^{t_2}[-\tfrac{1}{2}\rho DU^2 C_T h y' s_e - (m-\rho A)g y' s_e - \tfrac{1}{2}\rho D U C_N(\dot{y} + Uy')\bar{s}_e\\
&- [fM(\ddot{y} + U\dot{y}') + m\ddot{y}]s_e + fMU(\dot{y} + Uy')]\delta u_N\,\mathrm{d}t + \mathcal{O}(\epsilon^3), \qquad (64)
\end{aligned}
$$

where

$$
s_e = \frac{1}{A}\int_{L-l}^{L} A(s)\,\mathrm{d}s, \quad \bar{s}_e = \frac{1}{D}\int_{L-l}^{L} D(s)\,\mathrm{d}s, \qquad (65)
$$

and where it is understood that $A = A|_{L-l}$, $D = D|_{L-l}$, $y' = y'|_{L-l}$, $\dot{y} = \dot{y}|_{L-l}$, $h = D/D_h, M = \chi\rho A$. Typical values for s_e and $\bar{s}_e$, by way of illustration, are as follows. For a conical end, $D(s) = (D/l)(L-s), s_e = \frac{1}{3}l, \bar{s}_e = \frac{1}{2}l$; for a paraboloidal end, $D(s) = (D/\sqrt{l})\sqrt{L-s}, s_e = \frac{1}{2}l, \bar{s}_e = \frac{2}{3}l$; for an ellipsoidal end, $D(s) = D\{1-[(s-L+l)^2/l^2]\}^{1/2}, s_e = \frac{2}{3}l$ and $\bar{s}_e = \frac{\pi}{4}l$.

The first term in equation (64), in curly brackets, represents a small addendum to the axial pressure-tension terms in the equation of motion, rather than contributing to the boundary conditions. The second term, in square brackets, is associated with the transverse shear boundary condition. Accordingly, the boundary conditions at $s = L - l$ are (Lopes *et al.* 1999*a*):

$$
\begin{aligned}
-EIy''' + [fM(\ddot{y} + U\dot{y}') + m\ddot{y}]s_e - fMU(\dot{y} + Uy') + (m-\rho A)g y' s_e&\\
+\tfrac{1}{2}\rho DU^2 C_T h y' s_e + \tfrac{1}{2}\rho DUC_N(\dot{y} + Uy')\bar{s}_e = 0,& \qquad (66)
\end{aligned}
$$

and

$$
y'' = 0.
$$

The boundary conditions at $s = 0$ are, of course, $y(0) = 0$, $y'(0) = 0$.

Finally, equations (66) may be written in dimensionless form as follows:

$$
\begin{aligned}
-\eta''' + \chi_e[(1 + (\chi f - 1)\beta)\ddot{\eta} + \chi f\mathcal{U}\sqrt{\beta}\dot{\eta}'] + (\tfrac{1}{2}\bar{\chi}_e\varepsilon c_N - \chi f)(\mathcal{U}\sqrt{\beta}\dot{\eta} + \mathcal{U}^2\eta')&\\
+ (\tfrac{1}{2}\mathcal{U}^2\varepsilon c_T h + \gamma)\chi_e\eta' = \eta'' = 0 \text{ at } \xi = 1,& \qquad (67)
\end{aligned}
$$

where

$$
\chi_e = s_e/L, \quad \bar{\chi}_e = \bar{s}_e/L. \qquad (68)
$$

Other than the approximate empirical correlation between end-shape and f provided by Figures 8 and 9 of Part 1, there have been two attempts to determine f quantitatively. For conical and conoidal ends, Hannoyer & Païdoussis (1978) have proposed $f = 4s_e^2/[4s_e^2 + (D_o - D_i)^2] = 4\varepsilon^2\chi_e^2/[4\varepsilon^2\chi_e^2 + (1-\bar{\delta})^2]$, where $\bar{\delta} = D_i/D_o$ in cases where the cylinder is hollow and conveys fluid internally also, D_i and D_o being the inner and outer diameters of the cylinder; here $\bar{\delta} = 0$. A more elaborate method has been developed by Païdoussis & Yu (1976) for truncated ellipsoidal ends, which however does not provide an *explicit* expression for f. In the first case, it is presumed that separation does not occur at all, while in the second that it occurs exactly at the location where the ellipsoid is truncated.

7. METHODS OF ANALYSIS

The equation of motion (54) of the cantilevered cylinder, as derived in Section 5, is of third-order magnitude and hence nonlinear; furthermore, the boundary conditions (67) are time- and flow velocity-dependent. This renders the problem nonstandard and the solution procedure more complicated. As a first simplification, the partial differential equations (54) and (67) are transformed into a set of second-order ordinary differential equations using Galerkin's method. However, since the boundary conditions are time-dependent, different approaches are possible, as discussed in detail in Lopes *et al.* (1999*b*).[†]

Let us write for simplicity the equation of motion in the form $F(\eta, \mathscr{U}) = 0$. Then, with the boundary conditions added, the boundary value problem may be formulated as

$$F(\eta(\xi,\tau), \mathscr{U}) = 0, \tag{69}$$

$$\eta(0,\tau) = \eta'(0,\tau) = 0, \quad \eta''(1,\tau) = -\eta'''(1,\tau) + B(\eta(1,\tau), \mathscr{U}) = 0, \tag{70}$$

where $B(\eta, \mathscr{U})$ represents a complementary term in the end-shear boundary condition due to the tapering end. An alternative way of formulating the problem is the following:

$$F(\eta(\xi,\tau), \mathscr{U}) + \delta(\xi - 1)B(\eta(\xi,\tau), \mathscr{U}) = 0, \tag{71}$$

$$\eta(0,\tau) = \eta'(0,\tau) = 0, \quad \eta''(1,\tau) = \eta'''(1,\tau) = 0, \tag{72}$$

where $\delta(\xi - 1)$ is the Dirac delta function. With these two formulations in mind, three methods may be proposed to discretize the system, as follows.

Method (*a*) consists of utilizing the eigenfunctions $\Phi_j(\eta)$ of the problem $\eta'''' + \ddot{\eta} = 0$, i.e., the dry cantilevered-cylinder equation of motion, subject to boundary conditions (70), to discretize the system and apply them to the problem (69). In *Method* (*b*), the same eigenfunctions $\Phi_j(\xi)$ are used, but they are applied to an "expanded domain" of the problem, which effectively means that the time-dependent boundary condition, the last of equations (70), is added to the equation of motion, i.e., the expression $[-\eta'''(1,\tau) + B(\eta(1,\tau), \mathscr{U})]$ is added to the left-hand side of equation (69) via a Dirac delta function. Finally, in *Method* (*c*), the cantilever beam eigenfunctions $\phi_j(\xi)$ satisfying equations (72) are used directly to discretize equation (71).

It has been shown by Lopes *et al.* (1999*b*) that Method (b), although requiring only a small number of modes to yield extremely accurate results, is difficult to implement and is numerically very time-consuming, especially for a nonlinear problem. On the other hand, Method (c) is easy to implement and leads to more accurate results than Method (a), provided enough terms are used — cf. Païdoussis (1998, Section 4.6.2). Hence, this is the method that is presented here.

[†] An alternative would clearly be to use the Rayleigh–Ritz method.

This infinite-dimensional model is discretized by Galerkin's technique with the cantilever beam eigenfunctions, $\phi_j(\xi)$, used as a suitable set of base functions and with $q_j(\tau)$ the corresponding generalized coordinates; thus,

$$\eta(\xi,\tau) = \sum_{j=1}^{N} \phi_j(\xi) q_j(\tau), \tag{73}$$

where N represents the number of modes. Substituting expression (73) into (54), multiplying by $\phi_i(\xi)$ and integrating from 0 to 1, leads to the following matrix form:

$$\begin{aligned} &M_{ij}\ddot{q}_j + C_{ij}\dot{q}_j + K_{ij}q_j + r_{ijk}q_j|q_k| + \bar{s}_{ijk}|q_j|\dot{q}_k + \tilde{s}_{ijk}q_j|\dot{q}_k| + t_{ijk}\dot{q}_j|\dot{q}_k| \\ &+\alpha_{ijkl}q_jq_kq_l + \beta_{ijkl}q_jq_k\dot{q}_l + \gamma_{ijkl}q_j\dot{q}_k\dot{q}_l + \eta_{ijkl}\dot{q}_j\dot{q}_k\dot{q}_l + \mu_{ijkl}q_jq_k\ddot{q}_l = 0. \end{aligned} \tag{74}$$

Considering the linear terms, M_{ij}, C_{ij} and K_{ij} correspond to the mass, damping and stiffness matrices, respectively, while α_{ijkl}, β_{ijkl}, γ_{ijkl}, η_{ijkl}, μ_{ijkl}, r_{ijk}, $\bar{s}_{ijk}$, $\tilde{s}_{ijk}$, t_{ijk} are related to the nonlinear terms.

The mass, damping and stiffness matrices are defined by

$$M_{ij} = [1 + (\chi f - 1)\beta]\chi_e\phi_i(1)\phi_j(1) + [1 + (\chi - 1)\beta]\delta_{ij},$$

$$\begin{aligned} C_{ij} &= (\tfrac{1}{2}\bar{\chi}_e\varepsilon c_N - \chi f)\mathscr{U}\sqrt{\beta}\phi_i(1)\phi_j(1) + \chi f\mathscr{U}\sqrt{\beta}\chi_e\phi_i(1)\phi_j'(1) + 2\chi\mathscr{U}\sqrt{\beta}b_{ij} + \tfrac{1}{2}\mathscr{U}\varepsilon c_N\sqrt{\beta}\delta_{ij}, \\ K_{ij} &= (\gamma\chi_e + \tfrac{1}{2}\mathscr{U}^2(\varepsilon c_N\bar{\chi}_e + \varepsilon c_T h\chi_e) - \chi f\mathscr{U}^2)\phi_i(1)\phi_j'(1) + \chi\mathscr{U}^2 c_{ij} \\ &\quad + (\tfrac{1}{2}\mathscr{U}^2\varepsilon c_T(1+h) + \gamma)(d_{ij} - c_{ij}) + (\tfrac{1}{2}\mathscr{U}^2\varepsilon(c_N + c_T h) + \gamma)b_{ij} + \lambda_j^4\delta_{ij} - \tfrac{1}{2}\mathscr{U}^2 c_b c_{ij}, \end{aligned} \tag{75}$$

where the constants, b_{ij}, c_{ij}, d_{ij}, introduced by Païdoussis & Issid (1974), are defined by

$$b_{ij} = \int_0^1 \phi_i\phi_j'\,\mathrm{d}\xi, \quad c_{ij} = \int_0^1 \phi_i\phi_j''\,\mathrm{d}\xi, \quad d_{ij} = \int_0^1 \xi\phi_i\phi_j''\,\mathrm{d}\xi. \tag{76}$$

Furthermore, the nonlinear coefficients in equation (74) are defined by

$$\begin{aligned} \alpha_{ijkl} =& \tfrac{5}{2}\chi\mathscr{U}^2\int_0^1 \phi_i\phi_j''\phi_k'\phi_l'\,\mathrm{d}\xi + (\tfrac{1}{2}\mathscr{U}^2\varepsilon(c_N + c_T h) + \gamma)\,\tfrac{1}{2}\int_0^1 \phi_i\phi_j'\phi_k'\phi_l'\,\mathrm{d}\xi \\ &- (\tfrac{1}{2}\mathscr{U}^2\varepsilon c_T(1+h) + \gamma)\,\tfrac{3}{2}\int_0^1 (1-\xi)\phi_i\phi_j'\phi_k'\phi_l''\,\mathrm{d}\xi \\ &+ 4\int_0^1 \phi_i\phi_j'\phi_k''\phi_l'''\,\mathrm{d}\xi + \int_0^1 \phi_i\phi_j''\phi_k''\phi_l''\,\mathrm{d}\xi + \int_0^1 \phi_i\phi_j''''\phi_k'\phi_l'\,\mathrm{d}\xi \\ &- \chi\mathscr{U}^2\int_0^1 \phi_i\phi_j''\left(\int_\xi^1 \phi_k'\phi_l''\,\mathrm{d}\xi\right)\mathrm{d}\xi \\ &+ \tfrac{1}{2}\mathscr{U}^2 c_b\int_0^1 \phi_i\phi_j''\left(\int_\xi^1 \phi_k'\phi_l''\,\mathrm{d}\xi\right)\mathrm{d}\xi - \tfrac{1}{2}\mathscr{U}^2 c_b\int_0^1 \phi_i\phi_j'\phi_k'\phi_l''\,\mathrm{d}\xi \\ &+ \tfrac{1}{2}\mathscr{U}^2\varepsilon(c_T - c_N)\int_0^1 \phi_i\phi_j''\left(\int_\xi^1 \phi_k'\phi_l'\,\mathrm{d}\xi\right)\mathrm{d}\xi, \end{aligned}$$

$$\beta_{ijkl} = \chi \mathscr{U}\sqrt{\beta}\left\{\tfrac{7}{2}\cdot\int_0^1 \phi_i\phi_j'\phi_k'\phi_l'\,\mathrm{d}\xi - \tfrac{3}{2}\int_0^1 \phi_i\phi_j''\phi_k'\phi_l\,\mathrm{d}\xi - 2\int_0^1 \phi_i\phi_j''\left(\int_\xi^1 \phi_k'\phi_l'\,\mathrm{d}\xi\right)\mathrm{d}\xi\right.$$
$$\left. +2\int_0^1 \phi_i\phi_j''\left(\int_0^\xi \phi_k'\phi_l'\,\mathrm{d}\xi\right)\mathrm{d}\xi\right\} - \tfrac{1}{4}\mathscr{U}\sqrt{\beta}\varepsilon c_N\int_0^1 \phi_i\phi_j'\phi_k'\phi_l\,\mathrm{d}\xi$$
$$+\tfrac{1}{2}\mathscr{U}\sqrt{\beta}\varepsilon(c_T - c_N)\int_0^1 \phi_i\phi_j''\left(\int_\xi^1 \phi_k'\phi_l\,\mathrm{d}\xi\right)\mathrm{d}\xi,$$

$$\gamma_{ijkl} = -\tfrac{3}{2}\chi\beta\int_0^1 \phi_i\phi_j'\phi_k\phi_l'\,\mathrm{d}\xi - (1-\beta)\int_0^1 \phi_i\phi_j''\left(\int_\xi^1\int_0^\xi \phi_k'\phi_l'\,\mathrm{d}\xi\,\mathrm{d}\xi\right)\mathrm{d}\xi$$
$$+(1+(\chi-1)\beta)\int_0^1 \phi_i\phi_j'\left(\int_0^\xi \phi_k'\phi_l'\,\mathrm{d}\xi\right)\mathrm{d}\xi + 2\chi\beta\int_0^1 \phi_i\phi_k'\left(\int_0^\xi \phi_j'\phi_l'\,\mathrm{d}\xi\right)\mathrm{d}\xi$$
$$-\tfrac{1}{2}\beta\varepsilon c_N\int_0^1 \phi_i\phi_k\left(\int_0^\xi \phi_j'\phi_l'\,\mathrm{d}\xi\right)\mathrm{d}\xi - \tfrac{1}{4}\beta\varepsilon c_N\int_0^1 \phi_i\phi_j'\phi_k\phi_l\,\mathrm{d}\xi$$
$$+\tfrac{1}{4}\beta\varepsilon c_T\int_0^1 \phi_i\phi_j''\left(\int_\xi^1 \phi_k\phi_l\,\mathrm{d}\xi\right)\mathrm{d}\xi, \tag{77a}$$

$$\eta_{ijkl} = -\frac{1}{4}\frac{\beta^{3/2}\varepsilon c_N}{\mathscr{U}}\int_0^1 \phi_i\phi_j\phi_k\phi_l\,\mathrm{d}\xi,$$
$$\mu_{ijkl} = -(1-\beta)\int_0^1 \phi_i\phi_j''\left(\int_\xi^1\int_0^\xi \phi_k'\phi_l'\,\mathrm{d}\xi\,\mathrm{d}\xi\right)\mathrm{d}\xi - \chi\beta\int_0^1 \phi_i\phi_j''\left(\int_\xi^1 \phi_k'\phi_l\,\mathrm{d}\xi\right)\mathrm{d}\xi$$
$$+(1+(\chi-1)\beta)\int_0^1 \phi_i\phi_j'\left(\int_0^\xi \phi_k'\phi_l'\,\mathrm{d}\xi\right)\mathrm{d}\xi,$$

$$r_{ijk} = \tfrac{1}{2}\mathscr{U}^2\varepsilon c_d\textstyle\int_0^1 \phi_i\phi_j'|\phi_k'|\,\mathrm{d}\xi, \qquad \bar{s}_{ijk} = \tfrac{1}{2}\mathscr{U}\sqrt{\beta}\varepsilon c_d\int_0^1 \phi_i|\phi_j'|\phi_k\,\mathrm{d}\xi, \tag{77b}$$
$$\tilde{s}_{ijk} = \tfrac{1}{2}\mathscr{U}\sqrt{\beta}\varepsilon c_d\textstyle\int_0^1 \phi_i\phi_j'|\phi_k|\,\mathrm{d}\xi, \qquad t_{ijk} = \tfrac{1}{2}\beta\varepsilon c_d\int_0^1 \phi_i\phi_j|\phi_k|\,\mathrm{d}\xi.$$

8. CONCLUSION

In this paper, a nonlinear equation of motion, correct to $\mathcal{O}(\epsilon^3)$, has been derived for the dynamics of a cantilevered cylinder in axial flow via variational methods; in a consistent manner, linear boundary conditions have also been obtained for the case of the cylinder being terminated by a rigid, ogival end.

This equation is probably not *the* definitive nonlinear equation of motion for this system, since it was not obtained by a unified nonlinear treatment of the fluid mechanics. Nevertheless, the equation obtained provides a reasonable and useful tool for the exploration of the nonlinear dynamics of the system, which has hitherto been impossible. How successfully does this equation capture the true dynamics of the system has to be judged by comparing theoretical predictions with experimental observations. This is done in Part 3 of this study (Semler *et al.* 2002) and, as will be seen, agreement is reasonably good.

In the linear limit, the equation of motion and the boundary conditions are fundamentally the same as in Païdoussis (1973), though with some small improvements in the latter, notably including some viscous effects. Hence, agreement with linear aspects of observed behaviour (e.g., the threshold flow velocities for divergence and flutter) is expected to be similar to that displayed in Païdoussis (1973, 2002).

No conclusions as such can be drawn from the work so far, and the true conclusions relating to the theory are deferred to Part 3 of this work.

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APPENDIX A: THE EXPRESSIONS FOR F_{px} AND F_{py} IN LOPES *ET AL.* (1999*a*)

As mentioned in Section 4.2, in the original derivation, in Lopes *et al.* (1999*a*), the simplification was introduced that $\partial p/\partial x = \partial p/\partial X$.‡ The expressions for F_{px} and F_{py} then are as follows:

$$\begin{aligned} -F_{px} &= \tfrac{1}{2}y'^2\left(\tfrac{1}{2}\rho D U^2 C_T \frac{D}{D_h} - \rho g A\right) - y'y''Ap + \mathcal{O}(\epsilon^4), \\ F_{py} &= (y' - 2u'y' - \tfrac{1}{2}y'^3)\left(-\tfrac{1}{2}\rho D U^2 C_T \frac{D}{D_h} + \rho g A\right) \\ &\quad + (y'' - u''y' - 2u'y'' - \tfrac{3}{2}\,y'^2 y'')Ap + \mathcal{O}(\epsilon^5). \end{aligned} \qquad (A.1)$$

By comparing equation (A.1) to (45), it is seen that this "simplification" does not in fact result in expressions for F_{px} and F_{py} that are simpler, although the derivation was.

More important, however, is to note that when the correct expressions, equations (45), are used, the "symmetry" or parallelism between terms involving mg for the gravity components and $-\rho g A$ for the buoyancy components is achieved in the final equation of motion, which is absent in the Lopes *et al.* equation. Thus, in the corrected final dimensionless equations of motion, equation (52), it is possible to use a single parameter $\gamma = (m - \rho A)gL^3/EI$, while in Lopes *et al.* (1999*a,b*) we have to have two parameters separately, $\gamma_C = mg\,L^3/EI$ and $\gamma_F = \rho g A L^3/EI$.

The only reason for giving here the expressions for F_{px} and F_{py} in Lopes *et al.* is that these expressions, and the final equation incorporating them, have been used to conduct the calculations presented in Part 3. Nevertheless, discrepancies in the results only become important if γ is relatively large, i.e., if γ_C and γ_F are considerably different from each other, and h is large. However, in the calculations of Part 3, $\gamma = 0$ in some cases, while $\gamma = 1{\cdot}9$ in others; the results for $\gamma = 1{\cdot}9$ ($\gamma_C = 14{\cdot}4$ and $\gamma_F = 12{\cdot}5$) are virtually the same as for $\gamma = 0$. Furthermore, $h = 0$ has been taken throughout.

‡This, alternatively viewed, amounts to considering the elemental volume in equation (41) to be $A\delta x$.